BLOCH-KATO PRO-P GROUPS AND LOCALLY POWERFUL GROUPS

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To Professor Helmut Koch, with admiration on his 80th birthday.

Abstract. A Bloch-Kato pro-p group G is a pro-p group with the property that the $\mathbb{F}_p$-cohomology ring of every closed subgroup of G is quadratic. It is shown that either such a pro-p group G contains no closed free pro-p groups of infinite rank, or there exists an *orientation* $\theta\colon G \to \mathbb{Z}_p^\times$ such that G is θ-abelian. (See Thm B.) In case that G is also finitely generated, this implies that G is powerful, p-adic analytic with $d(G) = cd(G)$, and its $\mathbb{F}_p$-cohomology ring is an exterior algebra (see Cor. 4.8). These results will be obtained by studying locally powerful groups (see Thm A). There are certain Galois-theoretical implications, since Bloch-Kato pro-p groups arise naturally as maximal pro-p quotients and pro-p Sylow subgroups of absolute Galois groups (see Corollary 4.9). Finally, we study certain closure operations of the class of Bloch-Kato pro-p groups, connected with the Elementary type conjecture.

1. Introduction

Following [3] one calls a pro-p group G *a Bloch-Kato pro-p group* if the cohomology ring $H^\bullet(K, \mathbb{F}_p)$ is a quadratic $\mathbb{F}_p$-algebra for every closed subgroup K of G. From the positive solution of the Bloch-Kato conjecture recently obtained by M. Rost and V. Voevodsky (with C. Weibel's patch) one knows that for every field F containing a primitive pth root of unity the maximal pro-p quotient $G_F(p)$ of the absolute Galois group G_F of F is a Bloch-Kato pro-p group (see [24] and [25] for an overview of the proof, and [16], [26], [28], [29] for the foundation and completion of the proof).

The main goal of this paper is to estabilish a strong version of Tits alternative for Bloch-Kato pro-p groups. (See [23] or [6] for the original Tits alternative on linear groups.) For this purpose we study in section 3 *locally powerful* pro-p groups G, where we call a pro-p group G locally powerful if every finitely generated closed subgroup K of G is powerful. In order to state the classification of torsion-free, finitely generated, locally powerful pro-p groups effectively, we will introduce the notion of an *oriented pro-p group* (G, θ), i.e., G is a pro-p group and $\theta\colon G \to \mathbb{Z}_p^\times$ is a (continuous) homomorphism of pro-p groups, where $\mathbb{Z}_p$ denotes the ring of p-adic integers, and $\mathbb{Z}_p^\times \subset \mathbb{Z}_p$ denotes its group of units. For an oriented pro-p group (G, θ) one has a particular closed subgroup

$$\mathrm{Z}_\theta(G) = \left\{ h \in \ker(\theta) \,\middle|\, ghg^{-1} = h^{\theta(g)} \text{ for all } g \in G \right\}$$

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which will be called the *θ-center* of G. The oriented pro-p group (G,θ) will be called *θ-abelian*, if $\mathrm{Z}_\theta(G)=\ker(\theta)$. Thus every θ-abelian pro-p group is metabelian. Obviously, if $\theta\equiv\mathbf{1}$ is constant equal to 1, $\mathrm{Z}_{\mathbf{1}}(G)$ coincides with the center of G, and $(G,\mathbf{1})$ is $\mathbf{1}$-abelian if, and only if, G is abelian. In §3.4 we will prove the following theorem.

Theorem A. *A torsion-free finitely generated pro-p group G is locally powerful if, and only if, there exists an orientation $\theta\colon G\to\mathbb{Z}_p^\times$ such that (G,θ) is θ-abelian.*

Using Theorem A we will deduce in section 4 the following Tits alternative-type result for Bloch-Kato pro-p groups (see Theorem 4.6).

Theorem B. *Let p be an odd prime, and let G be a Bloch-Kato pro-p group. Then either G is θ-abelian for some orientation $\theta\colon G\to\mathbb{Z}_p^\times$ or G contains a closed non-abelian free pro-p subgroup.*

For p odd, a result similar to Theorem B was already proved by R. Ware for maximal pro-p Galois groups [27, Theorem 1 and Corollary 1]. His article was also a motivation for us to look for a Tits alternative in the class of Bloch-Kato pro-p groups.

In the final section we will consider free and direct products as well as inverse limits of Bloch-Kato pro-p groups. It will turn out that the class of Bloch-Kato pro-p groups is closed under free pro-p products (see Theorem 5.2), and under certain inverse limits (see Proposition 5.1). However, for direct products one has the following.

Theorem C. *Let G_1 and G_2 be Bloch-Kato pro-p groups, and assume that $G_1\times G_2$ is Bloch-Kato as well. Then the following restrictions hold:*

(i) *None of G_1 and G_2 is a powerful non-abelian Bloch-Kato group;*
(ii) *at least one of the two groups is abelian.*

Moreover, $\mathbb{Z}_p\times S$ is a Bloch-Kato pro-p group for any free pro-p group S.

The main reason for these last investigations is the connection with the Elementary Type conjecture for maximal pro-p Galois groups.

2. Preliminaries

We work in the category of pro-p groups. Henceforth subgroups are to be considered closed and all generators are to be considered topological generators (in the sense of the pro-p topology). For basic facts on Galois cohomology we refer to [18] or [20]. We abbreviate $H^k(G)$ for $H^k(G,\mathbb{F}_p)$ with the trivial G-action on $\mathbb{F}_p$. Thus $H^\bullet(G)=\bigoplus_{k\geq 0}H^k(G)$ denotes the graded cohomology ring equipped with the cup product $\cup$.

The first Bockstein homomorphism $\beta: H^1(G)\to H^2(G)$ is the connecting homomorphism arising from the short exact sequence of trivial G-modules

$$0\longrightarrow\mathbb{Z}/p.\mathbb{Z}\longrightarrow\mathbb{Z}/p^2.\mathbb{Z}\longrightarrow\mathbb{Z}/p.\mathbb{Z}\longrightarrow 0.$$

When $p=2$ one has $\beta(\chi)=\chi\cup\chi$ [4, Lemma 2.4].

If G is finitely generated, we denote by $d(G)$ the minimal number of generators of G, namely $d(G)=\dim(G/\Phi(G))$ as $\mathbb{F}_p$-vector space, where $\Phi(G)$ is the Frattini subgroup of G. In particular, if $d=d(G)$, we say that G is d-generated. Moreover, the rank $\mathrm{rk}(G)$ of G is $\sup\{d(K)|K\leq_c G\}$. If $G=S/R$ is a minimal presentation

for G, with S a free pro-p group such that $d(S) = d(G)$, then the relation rank $r(G)$ is the minimal number of generators of R as a closed normal subgroup of S. Moreover, it is well known that

$$d(G) = \dim_{\mathbb{F}_p}\left(H^1(G)\right) \quad \text{and} \quad r(G) = \dim_{\mathbb{F}_p}\left(H^2(G)\right)$$

(see [18, Ch. III §9]).

Finally, ${}^x y = xyx^{-1}$, and $[x,y] = {}^x y \cdot y^{-1}$ is the commutator of x and y, for $x, y \in G$.

As mentioned in the introduction, the maximal pro-p Galois group $G_F(p)$ of a field F containing the pth roots of unity μ_p is a Bloch-Kato group. Indeed if $\operatorname{char} F = p$ then $G_F(p)$ is a free pro-p group (see [20, II §2.2]), which is Bloch-Kato since a free pro-p group has cohomological dimension equal to 1.

Otherwise, for a profinite group G let $\mathcal{O}^p(G)$ be the subgroup

$$\mathcal{O}^p(G) = \langle K \in \mathrm{Syl}_\ell(G) | \ell \neq p \rangle,$$

where $\mathrm{Syl}_\ell(G)$ is the set of the Sylow pro-ℓ subgroups; namely $G/\mathcal{O}^p(G)$ is the maximal pro-p quotient of G [30, Proposition 2.1].

Let $F(p)$ be the maximal p-extension of a field F with $\operatorname{char} F \neq p$. Then the absolute Galois group of $F(p)$ is $G_{F(p)} = \mathcal{O}^p(G_F)$. Since $F(p)$ satisfies the hypotesis of the Bloch-Kato conjecture (i.e., $\mathrm{char}(F(p)) \neq p$ and $\mu_p \subseteq F(p)$), also the cohomology ring $H^\bullet(\mathcal{O}^p(G_F))$ is quadratic.

Moreover, as $\mathcal{O}^p(G_F)$ is p-perfect, $H^1(\mathcal{O}^p(G_F)) = 0$. Thus $H^\bullet(\mathcal{O}^p(G_F)) = 0$. This implies that in the Lyndon-Hochschild-Serre spectral sequence arising from $1 \to \mathcal{O}^p(G_F) \to G_F \to G_F(p) \to 1$ the terms

$$E_2^{rs} = H^r\left(G_F(p), H^s(\mathcal{O}^p(G_F))\right)$$

vanish for $s > 0$, and the spectral sequence collapses at the E_2-term. Hence the inflation map $H^n(G_F(p)) \to H^n(G_F)$ is an isomorphism for every $n \geq 0$ [18, Lemma 2.1.2]. Thus $H^\bullet(G_F(p))$ is quadratic as $H^\bullet(G_F)$ is quadratic by the Bloch-Kato conjecture.

Note that all the p-Sylow subgroups of an absolute Galois group – for any prime p – are Bloch-Kato pro-p groups (see [4, §9]).

Bloch-Kato pro-p groups have been defined and studied the first time in [3]. A fundamental feature of Bloch-Kato groups is the following: if p is odd then a Bloch-Kato pro-p group is *torsion-free* [3, Proposition 2.3], whereas the only non-trivial finite (pro-)2 Bloch-Kato groups are the elementary abelian 2-groups [3, Proposition 2.4].

If we keep in mind the Galois-theoretical background, this fact can be seen as an analogue of the celebrated Artin-Schreier theorem, which states that the only non-trivial finite subgroup of an absolute Galois group is C_2.

3. Locally powerful and oriented pro-p groups

3.1. **Powerful pro-p groups and Lie algebras.** A pro-p group G is said to be *powerful* if

$$[G,G] \subseteq \begin{cases} G^p & \text{for } p \text{ odd,} \\ G^4 & \text{for } p = 2, \end{cases}$$

where $[G,G]$ is the closed subgroup of G generated by the commutators of G, and G^p is the closed subgroup of G generated by the p-powers of the elements of G.

Let $\lambda_i(G)$ be the elements of the lower p-descending central series of the pro-p group G, namely $\lambda_1(G) = G$ and $\lambda_{i+1}(G) = \lambda_i(G)^p[\lambda_i(G), G]$. In particular, $\lambda_2(G)$ is the Frattini subgroup $\Phi(G)$. Then, a pro-p group G is called *uniformly powerful*, or simply *uniform*, if G is finitley generated, powerful, and

$$|\lambda_i(G) : \lambda_{i+1}(G)| = |G : \Phi(G)| \quad \text{for all } i \geq 1.$$

Thus a finitely generated powerful group is uniform if, and only if, it is torsion-free (see [7, Theorem 4.5]).

Recall that a pro-p group G is called locally powerful if every finitely generated closed subgroup K of G is powerful. Moreover, for uniform pro-p groups, one has the following property:

Proposition 3.1 ([7], Proposition 4.32)**.** *Let G be a d-generated uniform pro-p group, and let $\{x_1, \ldots, x_d\}$ be a generating set for G. Then G has a presentation $G = \langle x_1, \ldots, x_d | R\rangle$ with relations*

$$R = \left\{ [x_i, x_j] = x_1^{\lambda_1(i,j)} \cdots x_d^{\lambda_d(i,j)}, 1 \leq i < j \leq d \right\}, \tag{3.1}$$

and for all i, j one has $\lambda_n(i, j) \in p.\mathbb{Z}_p$ if p is odd, and $\lambda_n(i, j) \in 4.\mathbb{Z}_2$ if $p = 2$.

If G is a uniform pro-p group, then it is possible to associate a $\mathbb{Z}_p$-Lie algebra $L = \log(G)$ to it (see [7, §4.5] and [14]), i.e., L is the $\mathbb{Z}_p$-free module generated by the generators of G, equipped with the sum

$$x + y = \lim_{n\to\infty} x +_n y, \quad x +_n y = \left(x^{p^n} y^{p^n}\right)^{p^{-n}}, \tag{3.2}$$

and the Lie brackets

$$(x, y) = \lim_{n\to\infty} (x, y)_n, \quad (x, y)_n = \left[x^{p^n}, y^{p^n}\right]^{p^{-2n}}. \tag{3.3}$$

In analogy to pro-p groups we say that a $\mathbb{Z}_p$-Lie algebra L is *powerful* if $L \cong \mathbb{Z}_p^d$ for some $d > 0$ as $\mathbb{Z}_p$-module, and the derived algebra (LL) is contained in $p.L$ (resp. in $4.L$ if $p = 2$). It is well known that for a uniform group G the Lie algebra $\log(G)$ is powerful.

Remark 3.2.
(i) If $G = \langle x_1, \ldots, x_n\rangle$ is uniform, then it is possible to write every element $g \in G$ as $g = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$, with $\lambda_i \in \mathbb{Z}_p$, in a unique way. Thus the map

$$G \longrightarrow \log(G), \quad x_1^{\lambda_1} \cdots x_n^{\lambda_n} \longmapsto \lambda_1.x_1 + \ldots + \lambda_n.x_n$$

is a homeomorphism (in the $\mathbb{Z}_p$-topology) [7, Theorem 4.9].
(ii) If G is locally powerful and torsion-free, then every closed subfroup K of G is again a uniform group. Thus one can construct the Lie algebra $\log(K)$, which is in fact a subalgebra of $\log(G)$. In particular, the $\mathbb{Z}_p$-submodule $\mathrm{Span}_{\mathbb{Z}_p}\{x \in \Omega\}$ of $\log(G)$ is closed under Lie brackets for every subset $\Omega \subseteq G$.

3.2. **Oriented pro-p groups.** Let (G, θ), $\theta\colon G \to \mathbb{Z}_p^\times$, be an oriented pro-$p$ group. For every closed subgroup $K \leq_c G$, $(K, \theta|_K)$ is again an oriented pro-p. Notice that the image of θ is a pro-p subgroup of $\mathbb{Z}_p^\times$, thus $\mathrm{im}(\theta) \subseteq 1 + p.\mathbb{Z}_p$.

The following fact is straightforward.

Fact 3.3. *Let G be a pro-p group, and let $\theta,\theta'\colon G\to\mathbb{Z}_p^\times$, $\theta\neq\theta'$, be two distinct continuous homomorphisms such that G is both θ- and θ'-abelian. Then G is a closed subgroup of $\mathbb{Z}_p^\times$.*

The following property will turn out to be useful for our purpose.

Proposition 3.4. *Let (G,θ) be an oriented d-generated pro-p group G. Suppose further that* $\mathrm{im}(\theta)\leq_c 1+4.\mathbb{Z}_2$ *if $p=2$. Then G is θ-abelian if, and only if, there exists a presentation*

$$G=\left\langle x_1,\ldots,x_d \,\middle|\, [x_1,x_i]=x_i^\lambda, [x_i,x_j]=1, 2\leq i,j,\leq d\right\rangle, \tag{3.4}$$

where $\lambda=\theta(x_1)-1$ (if $p=2$ then $\lambda\in 4.\mathbb{Z}_2$).

Proof. Let G be θ-abelian, and put $A=\mathrm{im}(\theta)\leq_c 1+p.\mathbb{Z}_p$. By hypotesis, A is cyclic and torsion-free, i.e., either $A\cong\mathbb{Z}_p$ or $A=1$. In the latter case $G=\mathrm{Z}_\theta(G)$, namely, G is abelian. Otherwise one has the short exact sequence

$$1\longrightarrow \mathrm{Z}_\theta(G)\longrightarrow G\xrightarrow{\ \theta\ } A\longrightarrow 1,$$

which splits since $\mathbb{Z}_p$ is a projective pro-p group. This implies that $G\cong A\ltimes \mathrm{Z}_\theta(G)$, where the action of A on $\mathrm{Z}_\theta(G)$ is induced by θ. Therefore $A\ltimes\mathrm{Z}_\theta(G)$ has a presentation (3.4), where $d=d(\mathrm{Z}_\theta(G))+1$.

Conversely, suppose G is a pro-p group with presentation (3.4). Then one may construct an orientation $\theta\colon G\to\mathbb{Z}^\times$ such that $\theta(x_1)=1+\lambda$ and $\theta(x_i)=1$ for $i=2,\ldots,d$. Then $\mathrm{Z}_\theta(G)$ is generated by $x_2,\ldots,x_d$, and G is θ-abelian. □

3.3. **Oriented $\mathbb{Z}_p$-Lie algebras.** In analogy, we call a $\mathbb{Z}_p$-Lie algebra L together with a continuous homomorphism of Lie algebras $\theta_L\colon L\to\mathbb{Z}_p$, $\mathrm{im}(\theta_L)\subseteq p.\mathbb{Z}_p$, an *oriented $\mathbb{Z}_p$-Lie algebra*. Thus also in this case one may define the *θ_L-center* of L to be the ideal

$$\mathrm{Z}_{\theta_L}(L)=\{v\in\ker(\theta_L)\mid \mathrm{ad}\,x(v)=\theta_L(x).v \text{ for all } x\in L\}.$$

Then $\mathrm{Z}_{\theta_L}(L)$ is an abelian subalgebra of L. If $\mathrm{Z}_{\theta_L}(L)=\ker(\theta_L)$, then we call L a *θ_L-abelian $\mathbb{Z}_p$-Lie algebra.*

The following fact is straightforward.

Fact 3.5. *A $\mathbb{Z}_p$-Lie algebra L of rank d, together with an orientation θ_L, is θ_L-abelian if, and only if, L has a basis $\{v_i,\ldots,v_d\}$ such that $(v_1,v_i)=\lambda.v_i$ and $(v_i,v_j)=0$ for all $1<i,j\leq d$, where $\lambda=\theta_L(v_1)$ (if $p=2$ then $\lambda\in 4.\mathbb{Z}_2$).*

Combining Proprosition 3.4 and Fact 3.5, one obtains the following proposition.

Proposition 3.6. *A finitely generated uniform pro-p group G with orientation θ is θ-abelian if, and only if, the associated Lie algebra* $\log(G)$ *has an orientation θ_L such that* $\log(G)$ *is θ_L-abelian. In particular, $\theta_L=\log(\theta)$ and $\theta=\exp(\theta_L)$.*

Proof. From the construction of the Lie algebra $\log(G)$ given by (3.2) and (3.3), and from the presentation (3.4), computations show that if G is a uniform θ-abelian pro-p group then $\log(G)$ has Lie brackets as in Fact 3.5.

The map log from the category of uniform pro-p groups to the category of powerful Lie algebras over $\mathbb{Z}_p$ is a functor of categories. Moreover, the group structure can be reconstructed from the Lie algebra structure by the well known Baker-Campbell-Hausdorff series. Thus one has the functor exp from the category of

powerful $\mathbb{Z}_p$-Lie algebras to the category of uniform pro-p groups, which is the inverse of log. Namely log and exp are mutually inverse isomorphisms between the two categories [7, Theorem 9.10].

In particular, one has the following commutative diagram:

$$\begin{tikzcd} G \arrow[r, "\theta"] \arrow[d, bend left, "\log"] & \mathbb{Z}_p^\times \arrow[d, bend left, "\log"] \\ L \arrow[r, "\theta_L"'] \arrow[u, bend left, "\exp"] & p.\mathbb{Z}_p \arrow[u, bend left, "\exp"] \end{tikzcd}$$

this yields the claim. □

3.4. **Proof of Theorem A.**

Theorem A. *A finitely generated uniform pro-p group G is locally powerful if, and only if, G there exists an orientation $\theta\colon G \to \mathbb{Z}_p^\times$ such that (G, θ) is θ-abelian.*

Proof. If G is θ-abelian, then, by Proposition 3.4, G is locally powerful and torsion-free.

Conversely, let G be a torsion-free locally powerful pro-p group with $d(G) = d \geq 2$, Thus by Proposition 3.1, G has a presentation $G = \langle x_1, \ldots, x_d | R \rangle$ with relations as in (3.1). Let $H_{ij} \leq_c G$ be the closed subgroup generated by the elements x_i, x_j. Since H_{ij} is uniform as well, we have that

$$H_{ij} = \left\langle x_i, x_j \,\middle|\, [x_i, x_j] = x_i^{\lambda_i(i,j)} x_j^{\lambda_j(i,j)}, \lambda_i, \lambda_j \in p.\mathbb{Z}_p \right\rangle,$$

so that $R = \{[x_i, x_j] = x_i^{\lambda_i(i,j)} x_j^{\lambda_j(i,j)}, 1 \leq i < j \leq d\}$ is the set of relations.

Since an abelian pro-p group is $\mathbf{1}$-abelian, where $\mathbf{1}$ is the trivial orientation, we may assume that G is not abelian, i.e., we may assume without loss of generality that x_1 and x_2 do not commute.

Step 1: First suppose that $d = 2$. It is well known that if G is nonabelian, then G has a presentation $\langle x, y | [x, y] y^{-p^k} \rangle$ for some uniquely determined positive integer k [7, Chapter 4, Exercise 13]. Hence the claim follows from Fact 3.4.

Step 2: Suppose $d = 3$. By the previously mentioned remark we may choose x_1, x_2 such that $[x_1, x_2] = x_2^\lambda$, with $\lambda \in p.\mathbb{Z}_p$ (resp. $\lambda \in 4.\mathbb{Z}_2$ if $p = 2$). Thus

$$G = \left\langle x_1, x_2, x_3 \,\middle|\, [x_1, x_2] = x_2^\lambda, [x_1, x_3] = x_1^{\lambda_1} x_3^{\lambda_2}, [x_2, x_3] = x_2^{\mu_1} x_3^{\mu_2} \right\rangle,$$

with $\lambda_i, \mu_i \in p.\mathbb{Z}_p$ (resp. in $4.\mathbb{Z}_2$). Let H_{ij} be the subgroups as defined above, with $1 \leq i < j \leq 3$, and let $L = \log(G)$. Clearly, $(x_i, x_j)_n \in H_{ij}$ for all n. Hence $(x_i, x_j) \in \mathrm{Span}_{\mathbb{Z}_p}\{x_i, x_j\}$. In particular, the Lie brackets in L are such that

$$(x_1, x_2) = \alpha.x_2, \quad (x_2, x_3) = \beta_2.x_2 + \beta_3.x_3, \quad (x_1, x_3) = \gamma_1.x_1 + \gamma_3.x_3,$$

with $\alpha, \beta_i, \gamma_i \in p.\mathbb{Z}_p$ (resp. in $4.\mathbb{Z}_2$).

By the Jacobi identity, one has

$$\begin{aligned} 0 &= ((x_1, x_2), x_3) + ((x_2, x_3), x_1) + ((x_3, x_1), x_2) \\ &= (\alpha.x_2, x_3) + (\beta_2.x_2 + \beta_3.x_3, x_1) - (\gamma_1.x_1 + \gamma_3.x_3, x_2) \\ &= -\beta_3\gamma_1.x_1 + (\alpha\beta_2 - \alpha\beta_2 - \alpha\gamma_1 + \beta_2\gamma_3)\,.x_2 + (\alpha\beta_3 - \beta_3\gamma_3 + \beta_3\gamma_3)\,.x_3, \end{aligned}$$

hence $\beta_3\gamma_1.x_1 = 0$, and thus $\beta_3 = 0$ or $\gamma_1 = 0$.

(1) If $\beta_3 = 0$, then by definition $(x_2, x_3) \in \mathrm{Span}_{\mathbb{Z}_p}\{x_2\}$, i.e., $\mathrm{Span}_{\mathbb{Z}_p}\{x_2\}$ is an ideal of L. Therefore we may choose without loss of generality x_1 and x_3 such that $(x_1, x_3) \in \mathrm{Span}_{\mathbb{Z}_p}\{x_3\}$, and $(x_i, x_2) \in \mathrm{Span}_{\mathbb{Z}_p}\{x_2\}$ for $i = 1, 3$.

(2) If $\gamma_1 = 0$, then by definition $(x_1, x_3) \in \mathrm{Span}_{\mathbb{Z}_p}\{x_3\}$, i.e., $\mathrm{Span}_{\mathbb{Z}_p}\{x_2, x_3\}$ is an ideal of L. Therefore we may choose without loss of generality x_2 and x_3 such that $(x_2, x_3) \in \mathrm{Span}_{\mathbb{Z}_p}\{x_2\}$, and $(x_1, x_i) \in \mathrm{Span}_{\mathbb{Z}_p}\{x_i\}$ for $i = 2, 3$.

Altogether the Lie brackets in L are

$$(x_1, x_2) = \alpha'.x_2, \quad (x_2, x_3) = \beta'.x_2, \quad (x_1, x_3) = \gamma'.x_3,$$

with $\alpha', \beta', \gamma' \in p.\mathbb{Z}_p$ (resp. in $4.\mathbb{Z}_2$). The matrix of $\mathrm{ad}(\gamma.x_3)$ with respect to the basis $\{x_1, x_2, x_3\}$ is given by

$$\mathrm{ad}(\gamma'.x_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta'\gamma' & 0 \\ -\gamma'^2 & 0 & 0 \end{pmatrix}.$$

In particular, its trace is $\mathrm{tr}(\mathrm{ad}(\gamma'.x_3)) = \beta'\gamma'$. Since $\mathrm{ad}(\gamma'.x_3) = (\mathrm{ad}(x_1), \mathrm{ad}(x_3))$, one has $\mathrm{tr}(\mathrm{ad}(\gamma'.x_3)) = \beta'\gamma' = 0$. Therefore $\beta' = 0$ or $\gamma' = 0$.

(1) If $\beta' = 0$, let $v_1 = x_1 + x_2$ and $v_2 = x_2 + x_3$. Then $(v_1, v_2) = \alpha'.x_2 + \gamma'.x_3$. By Remark 3.2, one has that $(v_1, v_2) \in \mathrm{Span}_{\mathbb{Z}_p}\{v_1, v_2\}$. Thus (v_1, v_2) is necessarily a multiple of v_2, i.e., $\alpha' = \gamma'$.

(2) If $\gamma' = 0$ and $\beta' \neq 0$, let $v = x_1 + x_2$. Then $(v, x_3) = \beta'.x_2$. By Remark 3.2, one has that $(v, x_3) \in \mathrm{Span}_{\mathbb{Z}_p}\{v, x_3\}$. In particular, no multiple of x_2 lies in $\mathrm{Span}_{\mathbb{Z}_p}\{v, x_3\}$. Therefore, this case is impossible.

(3) If $\beta' = \gamma' = 0$ then $\alpha' = 0$ by (1). So L, and hence G, is abelian. But this case was excluded.

This yields $\beta' = 0$ and $\alpha' = \gamma' \neq 0$, with $\alpha' \in p.\mathbb{Z}_p$ (resp. in $4.\mathbb{Z}_2$). Therefore, by Fact 3.4 (ii), L is θ_L-abelian, with $\theta_L(x_1) = \alpha'$, $\theta_L(x_i) = 0$ for $i = 2, 3$, and the claim follows from Proposition 3.6.

Step 3: Finally, suppose that G is locally powerful, torsion-free with $d(G) = n + 1 \geq 4$, and let G be generated by $x_1, \ldots, x_{n+1}$. Since G is non-abelian we may assume without loss of generality that x_1 and x_2 do not commute.

Let $H \leq_c G$ be the subgroup generated by $x_1, \ldots, x_n$. Thus by induction there is a unique (non-trivial) orientation $\theta \colon H \to \mathbb{Z}_p^\times$ such that H is θ-abelian. In particular, we may assume that $[x_1, x_i] = x_i^\lambda$ and $[x_i, x_j] = 1$ for all $2 \leq i, j \leq n$, where $\lambda = \theta(x_1) - 1 \in p.\mathbb{Z}_p \smallsetminus \{0\}$ (resp. in $4.\mathbb{Z}_2 \smallsetminus \{0\}$ for $p = 2$).

Furthermore, let $H_i \leq_c G$ be the subgroup generated by x_1, x_i, x_{n+1}, for $2 \leq i \leq n$. By induction, for each i there exists an orientation $\theta_i \colon H_i \to \mathbb{Z}_p^\times$ such that H_i is θ_i-abelian.

Since $\theta_i(x_1) = \theta(x_1) = 1 + \lambda$ and $\theta_i(x_i) = \theta(x_i) = 1$ for all i, then necessarily $\theta_i(x_{n+1}) = 1$ for all i; i.e., $[x_1, x_{n+1}] = x_{n+1}^\lambda$ and $[x_i, x_{n+1}] = 1$ for all i. Hence we may extend θ to G such that $\theta(x_{n+1}) = 1$. Thus G is θ-abelian.

This estabilishes the theorem. □

4. A Tits alternative for Bloch-Kato pro-p groups

4.1. **Dimension of cohomology groups.** If $p = 2$ then the cohomology ring of a Bloch-Kato group G is a quotient of the symmetric algebra $S^\bullet(H^1(G))$. On the

other hand, if p is odd then the cohomology ring of G is a quotient of the exterior algebra $\bigwedge_\bullet(H^1(G))$. Thus in this latter case if G is finitely generated then

$$\dim_{\mathbb{F}_p}(H^r(G)) \leq \binom{d(G)}{r} \quad \text{for all } r \geq 0. \tag{4.1}$$

In fact it is possible to prove a stronger result.

Proposition 4.1. *Let p be odd, and let G be a finitely generated Bloch-Kato pro-p group. Then*

(i) $cd(G) \leq d(G) \leq \dim_{\mathbb{F}_p}(\lambda_2(G)/\lambda_3(G))$;
(ii) $r(G) \leq \binom{d(G)}{2}$.

Proof. The inequalities $cd(G) \leq d(G)$ and $r(G) \leq \binom{d(G)}{2}$ are immediate consequences of (4.1).

The inflation map induces an isomorphism $\rho = \inf^1_{G/\Phi(G)}$ in degree 1, so that the commutativity of the diagram

$$\begin{tikzcd} H^1(G/\Phi(G)) \otimes H^1(G/\Phi(G)) \arrow[r, "\cup"] \arrow[d, "\rho\otimes\rho", "\wr"'] & H^2(G/\Phi(G)) \arrow[d, "\inf^2_{G/\Phi(G)}"] \\ H^1(G) \otimes H^1(G) \arrow[r, "\cup", two heads] & H^2(G) \end{tikzcd}$$

implies that $\inf^2_{G/\Phi(G)}$ is surjective.

Consider the five terms exact sequence arising from the quotient $G/\Phi(G)$. Since ρ is an isomorphism, it reduces to

$$0 \longrightarrow H^1(\Phi(G))^G \longrightarrow H^2(G/\Phi(G)) \xrightarrow{\inf^2_{G/\Phi(G)}} H^2(G) \longrightarrow 0. \tag{4.2}$$

Moreover, the group $H^1(\Phi(G))^G$ is isomorphic to the quotient $(\lambda_2(G)/\lambda_3(G))^*$ as discrete group, where $\_^*$ denotes the Pontryagin dual.

Since $G/\Phi(G)$ is a elementary abelian p-group, the second cohomology group is

$$\begin{aligned} H^2(G/\Phi(G)) &= \beta\left(H^1(G/\Phi(G))\right) \oplus \left(H^1(G/\Phi(G)) \cup H^1(G/\Phi(G))\right) \\ &\cong H^1(G/\Phi(G)) \oplus \left(H^1(G/\Phi(G)) \wedge H^1(G/\Phi(G))\right) \\ &\cong H^1(G) \oplus \left(H^1(G) \wedge H^1(G)\right). \end{aligned}$$

From the sequence (4.2) one obtains

$$0 \longrightarrow (\lambda_2(G)/\lambda_3(G))^* \longrightarrow H^1(G) \oplus \left(H^1(G) \wedge H^1(G)\right) \longrightarrow H^2(G) \longrightarrow 0. \tag{4.3}$$

Therefore

$$\begin{aligned} d(G) + \binom{d(G)}{2} &= \dim_{\mathbb{F}_p}\left(H^1(G) \oplus \left(H^1(G) \wedge H^1(G)\right)\right) \\ &= \dim_{\mathbb{F}_p}\left(\frac{\lambda_2(G)}{\lambda_3(G)}\right) + \dim_{\mathbb{F}_p}\left(H^2(G)\right) \quad \text{by (4.3)} \\ &= \dim_{\mathbb{F}_p}\left(\frac{\lambda_2(G)}{\lambda_3(G)}\right) + r(G) \\ &\leq \dim_{\mathbb{F}_p}\left(\frac{\lambda_2(G)}{\lambda_3(G)}\right) + \binom{d(G)}{2}, \end{aligned}$$

namely $d(G) \leq \dim_{\mathbb{F}_p}(\lambda_2(G)/\lambda_3(G))$. □

Remark 4.2. There is no analogue of Proposition 4.1 in case that $p = 2$. For a Bloch-Kato pro-2 group G the exact sequence (4.2) specifies to

$$0 \longrightarrow (\lambda_2(G)/\lambda_3(G))^* \longrightarrow H^2\left((\mathbb{Z}/2\mathbb{Z})^d\right) \longrightarrow H^2(G) \longrightarrow 0,$$

and $\dim(H^2((\mathbb{Z}/2\mathbb{Z})^d)) = \binom{d+1}{2}$, while $\dim(H^2(G)) \leq \binom{d+1}{2}$.

Proposition 4.3. *Let p be odd, and let G be a Bloch-Kato pro-p group such that $cd(G) = d(G)$. Then the cohomology ring $H^\bullet(G)$ is isomorphic to the $\mathbb{F}_p$-exterior algebra $\bigwedge_\bullet(H^1(G))$.*

Proof. Let $H^1(G)$ be freely generated by $\chi_1, \ldots, \chi_d$ as $\mathbb{F}_p$ vector space, and suppose for contradiction that $H^\bullet(G)$ is a non-trivial quotient of $\bigwedge_\bullet(H^1(G))$. Since $H^\bullet(G)$ is quadratic, there is a non-trivial relation in $H^1(G) \wedge H^1(G)$. Thus we may assume without loss of generality that

$$\chi_1 \cup \chi_2 = \sum_{(i,j)\neq(1,2)} a_{ij}.\chi_i \cup \chi_j,$$

with $i < j$ and $a_{ij} \in \mathbb{F}_p$. This implies that

$$\chi_1 \cup \chi_2 \cup \cdots \cup \chi_d = \sum_{(i,j)\neq(1,2)} a_{ij}.\chi_i \cup \chi_j \cup \chi_3 \cup \cdots \cup \chi_d = 0,$$

namely $H^d(G) = \mathrm{Span}_{\mathbb{F}_p}\{\chi_1 \cup \cdots \cup \chi_d\} = 0$, a contradiction. This yields the claim. □

4.2. **Powerful groups and the cup product.** The following theorem is due to P. Symonds and Th. Weigel:

Theorem 4.4 ([21], Theorem 5.1.6)**.** *Let G be a finitely generated pro-p group. Then the map*

$$\Lambda_2(\cup)\colon H^1(G) \wedge H^1(G) \longrightarrow H^2(G)$$

induced by the cup product is injective if, and only if, G is powerful.

Let G be a pro-p group, and let H be a closed subgroup of G. Then we call H *properly embedded* in G, if the canonical map $H/\Phi(H) \to G/\Phi(G)$ is injective. The following fact is a direct consequence of Pontryagin duality.

Fact 4.5. *Let G be a pro-p group, and let H be a closed subgroup of G. Then the following are equivalent.*

(i) *H is properly embedded in G.*
(ii) $\mathrm{res}^1_{G,H}\colon H^1(G,\mathbb{F}_p) \to H^1(H,\mathbb{F}_p)$ *is surjective.*

Theorem 4.6. *Let p be an odd prime, and let G be a Bloch-Kato pro-p group. Then the following are equivalent:*

(i) *G does not contain non-abelian closed free pro-p subgroups.*
(ii) *G is locally powerful.*
(iii) *there exists an orientation $\theta\colon G \to \mathbb{Z}_p^\times$ such that (G,θ) is θ-abelian. In particular, G is metabelian.*

Moreover, if G is finitely generated, then (i)*,* (ii)*, and* (iii) *are equivalent to:*

(iv) *G is p-adic analytic.*

Proof. Suppose that (i) holds and that G is not locally powerful. Then there exists a finitely generated subgroup $K \leq_c G$ which is not powerful. In particular, the map

$$\Lambda_2(\cup)\colon H^1(K) \wedge H^1(K) \longrightarrow H^2(K)$$

is not injective. Let $\chi_1, \ldots, \chi_r$ be an $\mathbb{F}_p$-basis of the $\mathbb{F}_p$-vector space $H^1(K)$. Thus there exists a non-trivial element

$$\eta = \sum_{1\leq i<j\leq r} a_{ij}.\chi_i \wedge \chi_j \in \ker\left(\Lambda_2(\cup)\right).$$

As $\eta \neq 0$, there exist $m, n \in \{1, \ldots, r\}$, $m < n$, such that $a_{mn} \neq 0$. Let $x_1, \ldots, x_r \in K$ be a minimal generating system of K satisfying $\chi_i(x_j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, r\}$, and let $S = \langle x_m, x_n\rangle$. Then S is properly embedded in K, $\rho = \mathrm{res}^1_{K,S}\colon H^1(K) \to H^1(S)$ is surjective, and, by construction, $\ker(\rho) = \mathrm{Span}_{\mathbb{F}_p}\{\, \chi_i \mid 1 \leq i \leq r,\, i \neq n, m \,\}$. From the surjectivity of $\rho \wedge \rho$ and the commutativity of the diagram

$$\begin{tikzcd} H^1(K) \wedge H^1(K) \arrow[r, "\Lambda_2(\cup)"] \arrow[d, "\rho\wedge\rho"'] & H^2(K) \arrow[d, "\mathrm{res}^2_{K,S}"] \\ H^1(S) \wedge H^1(S) \arrow[r, "\Lambda_2(\cup)"] & H^2(S) \end{tikzcd}$$

one concludes that the map $\Lambda_2(\cup)\colon H^1(S) \wedge H^1(S) \to H^2(S)$ is the 0-map. Thus – as S is Bloch-Kato – $H^2(S) = 0$, i.e., S is a 2-generated free pro-p group (see [18, Proposition 3.5.17]), a contradiction. This shows that (i) implies (ii).

The implication (ii) $\Rightarrow$ (i) follows from the fact that a free pro-p group which is powerful must be cyclic. Moreover, the equivalence (ii) $\Leftrightarrow$ (iii) follows from Theorem A. If G is finitely generated, the implication (ii) $\Rightarrow$ (iv) is well known (see [7, Theorem 8.18]), whereas the implication (iv) $\Rightarrow$ (i) follows from [7, Theorem 8.32]. This yields the claim. □

Remark 4.7. Notice that in the proof we do not require the group G to be Bloch-Kato; in fact it is enough to assume that the cohomology of every closed subgroup of G is decomposable, i.e., it is generated in degree one (thus G is *almost* Bloch-Kato, in the language of [3]).

Corollary 4.8. *Let p be an odd prime, and let G be a Bloch-Kato pro-p group. Then the following are equivalent*

(i) *G is powerful.*
(ii) *G contains no free pro-p groups of infinite rank.*
(iii) *There exists an orientation $\theta : G \to \mathbb{Z}_p^\times$ such that G is θ-abelian.*

Furthermore, if G is finitley generated, these properties are equivalent to

(iv) *G is p-adic analytic.*
(v) *$cd(G) = d(G)$.*
(vi) *$H^\bullet(G) \cong \bigwedge_\bullet ((G/\Phi(G))^*)$.*

As we stressed in §2, Bloch-Kato groups arise naturally as maximal pro-p Galois groups and p-Sylow subgroups of absolute Galois groups. Thus the above results provide strong restrictions to such groups. In particular, one obtains the following result:

Corollary 4.9. *Let F be a field, such that $G_F(p)$ is a metabelian pro-p group (i.e., the commutator subgroup of $G_F(p)$ is abelian). If $F \supseteq \mu_p$ then $G_F(p)$ has generators $\{\sigma, \rho_i\}_{i\in\mathcal{I}}$ with relations $[\rho_i, \rho_j] = 1$ and $\rho_i^\sigma = \rho_i^{q+1}$, where $q = 0$ if $\mu_{p^k} \subseteq F$ for all $k \geq 1$, or $q = p^n$, where n is the largest integer such that $F \supseteq \mu_{p^n}$.*

This corollary provides the answer to a question raised by R. Ware in his paper [27, page 727]. Indeed he managed to prove that $G_F(p)$ has such a presentation if F contains also a p^2th root of unity (and not only a pth root), though it seemed reasonable that such an assumption is not necessary – as, in fact, it is not.

In this case, the suitable orientation θ of $G_F(p)$ is the cyclotomic character, i.e., the map

$$\theta\colon G_F(p) \longrightarrow \mathrm{End}_F(\mu_{p^\infty}) \cong \mathbb{Z}_p^\times,$$

where $\mu_{p^\infty} \leq \bar{F}^{sep}$ denotes the group of roots of unity of p-power order.

In particular, the θ-center is $\mathrm{Z}_\theta(G_F(p)) = G_L(p)$, where $L = F(\mu_{p^\infty})$.

Example 4.10. Let $q = p^n$ be a (non-trivial) p-power, and let F be the field $F = k((\mathfrak{X}))$, where $k = \mathbb{F}_\ell(\mu_q)$, with $\ell \equiv 1 \bmod p$, and $\mathfrak{X} = \{X_1, \ldots, X_n\}$. Then $G_F(p)$ has generators $\{\sigma, \rho_i\}_{i=1}^n$ with relations $[\rho_i, \rho_j] = 1$ and $\rho_i^\sigma = \rho_i^{q+1}$. Furhtermore, if $\mu_q \subseteq k$ for every p-power q, then $G_F(p)$ is abelian, i.e., $G_F(p) \cong \mathbb{Z}_p^n$.

The case $p = 2$ is more subtle, since the pro-2 version for Theorem 4.4 is more involuted. Thus it turns out that it is impossible to state Theorem B also for Bloch-Kato pro-2. For example the pro-2 dihedral group

$$C_2 \ltimes \mathbb{Z}_2(2) = \left\langle \sigma, \rho \,\middle|\, \sigma^2 = 1, {}^\sigma\rho = \rho^{-1} \right\rangle$$

is θ-abelian, with $\theta(\sigma) = -1$, $\theta(\rho) = 1$, and it contains no non-abelian closed free pro-2 subgroups, yet it is not powerful.

Nevertheless, it is possible to get a similar result when we add more restrictions to G, and using [31, Theorem C].

Theorem 4.11. *Let G be a Bloch-Kato pro-2 group such that G is torsion-free, and assume that the first Bockstein homomorphism $\beta : H^1(G) \to H^2(G)$ is trivial. Then the following are equivalent:*

(i) *Every non-trivial closed free subgroup of G is cyclic.*
(ii) *G is locally powerful.*
(iii) *there exists an orientation $\theta\colon G \to \mathbb{Z}_2^\times$ such that (G, θ) is θ-abelian.*

5. The class of Bloch-Kato pro-p groups

Some time ago I. Efrat has formulated a conjecture – the so called "elementary type conjecture" – for maximal pro-p Galois groups, which states that the group structure of maximal pro-p Galois groups of some fields is very restricted, namely such groups are free pro-p products and semidirect products of certain pro-p groups (see [9], [12]).

It seems very difficult to decide whether such an "elementary type" conjecture should hold already for the class of finitely generated Bloch-Kato pro-p groups. All known examples of Bloch-Kato pro-p groups have this property, but apart from this fact there is little evidence.

For this reason we investigate certain closure operations for the class of Bloch-Kato pro-p groups.

5.1. **Projective limits and free products of Bloch-Kato groups.**

Proposition 5.1. *Let $\{G_i, \pi_{ij}\}_{i\in I}$ be projective system of Bloch-Kato pro-p groups with π_{ij} surjective for all $i \leq j$, such that the maps*

$$\inf_{ij}^{\bullet} : H^{\bullet}(G_j) \to H^{\bullet}(G_i)$$

induced by $\pi_{ij} : G_j \to G_i$ are injective for any $i \leq j$. Then for $\hat{G} = \varprojlim_i G_i$, the cohomology ring $H^{\bullet}(\hat{G})$ is quadratic.

Proof. It is well known that

$$\varinjlim_{i\in I} H^n(G_i) \cong H^n(\hat{G})$$

for every $n \geq 0$ [18, Proposition 1.5.1].

Moreover, the class of quadratic $\mathbb{F}_p$-algebras is closed under certain direct limits: namely if $A_{\bullet}^i$ is a quadratic $\mathbb{F}_p$-algebra for all $i \geq 0$ with $A_{\bullet} = \varinjlim_i A_{\bullet}^i$ and such that the maps $A_n^i \to A_n^j$ are injective for all $i \leq j$, then $A_{\bullet}$ is quadratic. This implies that $H^{\bullet}(\hat{G})$ is quadratic. □

In order to state and prove the following theorem, we need O. Mel'nikov's version of the Kurosh subgroup theorem for free pro-p products (see [17]).

Let T be a profinite space, and let $\{G_t\}_{t\in T}$ be a family of pro-p groups. Then such a family defines a *sheaf* $\mathcal{G}$ of pro-p groups, i.e., a profinite space $\mathcal{G}$ together with a continuous surjection $\gamma : \mathcal{G} \to T$ such that for all $t \in T$, $\gamma^{-1}(t) = G_t$, and the group operation of G_t depends continuously on t. The free pro-p product of the family $\{G_t\}$ is the pro-p group $G = \coprod_t G_t$ together with a morphism $\iota\colon \mathcal{G} \to G$ such that for any pro-p group H and for any continuous map $\varphi\colon \mathcal{G} \to H$ whose restrictions $\varphi|_{G_t}\colon G_t \to H$ are all homomorphisms of pro-p groups, there exists a unique homomorphism $\widetilde{\varphi}\colon G \to H$ such that $\widetilde{\varphi} \circ \iota = \varphi$.

Theorem 5.2. *Let $G = \coprod_t G_t$ be the free product in the category of pro-p groups of a family of Bloch-Kato pro-p groups $\{G_t\}_{t\in T}$, where T is a profinite space. Then G is a Bloch-Kato pro-p group. In particular, the free pro-p product of two Bloch-Kato pro-p groups G_1 and G_2 is a Bloch-Kato pro-p group.*

Proof. Let K be a closed subgroup of G. Then by [17, Theorem 4.3] it is possible to decompose K in the following way:

$$K = \left(\coprod_{t\in T} \left(\coprod_{K\backslash G/G_t} (K \cap G_t^r) \right) \right) \sqcup S,$$

where S is a free pro-p group and the r vary over a set $\mathcal{R}_t \subset G$ of representatives of the coset space $K\backslash G/G_t$ – which is profinite.

In particular, K is the free pro-p product (over a profinite set) of closed subgroups of the groups G_t. Let $K_t = \coprod_{r\in\mathcal{R}_t}(K \cap G_t^r)$, so that $K = S \sqcup (\coprod_t K_t)$. As a consequence of [17, Theorems 4.1 and 4.2], one has that the homology corestriction maps

$$\operatorname{cor}_n^{K_t}\colon \bigoplus_{K\backslash G/G_t} H_n\left(K \cap G_t^r, \mathbb{F}_p\right) \longrightarrow H_n\left(K_t, \mathbb{F}_p\right), \quad \text{and}$$

$$\operatorname{cor}_n^{K}\colon \bigoplus_{t\in T} H_n\left(K_t, \mathbb{F}_p\right) \longrightarrow H_n\left(K, \mathbb{F}_p\right)$$

are isomorphisms for $n \geq 1$. By Pontryagin duality, i.e.,

$$H_\bullet(G, \mathbb{F}_p)^* \cong H^\bullet\left(G, \mathbb{F}_p^*\right),$$

also the cohomology restriction maps

$$\operatorname{res}^n_{K_t} \colon H^n(K_t) \longrightarrow \bigoplus_{K\backslash G/G_t} H^n(K \cap G_t^r), \quad \text{and} \tag{5.1}$$

$$\operatorname{res}^n_{K} \colon H^n(K) \longrightarrow \bigoplus_{t \in T} H^n(K_t) \tag{5.2}$$

are isomorphisms for $n \geq 1$. Since the groups G_t are Bloch-Kato pro-p groups, the cohomology rings $H^\bullet(K \cap G_t^r)$ are quadratic. Thus, by (5.1) and (5.2) also the cohomology rings $H^\bullet(K_t)$ and $H^\bullet(K)$ are quadratic. Therefore G is Bloch-Kato. □

5.2. **Direct products of Bloch-Kato groups.** It seems natural to consider direct products of Bloch-Kato pro-p groups. The next result is a consequence of Theorem A:

Proposition 5.3. *The direct product of a powerful non-abelian Bloch-Kato group G with any pro-p group is not Bloch-Kato.*

Proof. Let $\widetilde{G} = G \times \mathbb{Z}_p$, with G θ-abelian, but not abelian, and suppose $\widetilde{G}$ is Bloch-Kato. Since $\widetilde{G}$ contains no free pro-p groups of rank greater than 1, $\widetilde{G}$ must be $\widetilde{\theta}$-abelian, for some orientation $\widetilde{\theta} \colon \widetilde{G} \to \mathbb{Z}_p^\times$ such that $\widetilde{\theta}|_G = \theta$. But the action of G on $\mathbb{Z}_p$ is trivial, and G is non-abelian (i.e., θ is not trivial), thus $\mathbb{Z}_p \nleq \mathrm{Z}_\theta(G)$, and $\widetilde{G}$ cannot be $\widetilde{\theta}$-abelian, a contradiction. Therefore $\widetilde{G}$ is not Bloch-Kato, and this implies that the direct product of a powerful non-abelian Bloch-Kato group with any non-trivial pro-p group is not Bloch-Kato. □

However, one has the following.

Theorem 5.4. *Let S be a free pro-p group, and let $\widetilde{G} = \mathbb{Z}_p \times S$. Then $\widetilde{G}$ is a Bloch-Kato pro-p group.*

Proof. First of all we show that the cohomology ring $H^\bullet(\widetilde{G})$ is a quadratic $\mathbb{F}_p$-algebra. By [18, Theorem 2.4.6], one has that

$$H^n\left(\widetilde{G}\right) = H^n(S) \oplus H^{n-1}(S), \quad \text{for } n \geq 1.$$

In particular, let $\{x_i\}_{i\in I}$ be a set of free generators of S, and let y be a generator of $\mathbb{Z}_p$. Moreover, let $x_i^* \in H^1(S)$ be the Pontryagin dual of x_i, for all $i \in I$, and let $y^* \in H^1(\mathbb{Z}_p)$ be the dual of y. Then

$$H^\bullet\left(\widetilde{G}\right) = \mathbb{F}_p \oplus \operatorname{Span}_{\mathbb{F}_p}\{x_i^*, y^*\}_{i\in I} \oplus \operatorname{Span}_{\mathbb{F}_p}\{x_i^* \cup y^*\}_{i\in I},$$

namely, $H^\bullet(\widetilde{G})$ is a quadratic $\mathbb{F}_p$-algebra.

Let K be a closed subgroup of $\widetilde{G}$, and put $N = K \cap S$. Then N is a free pro-p group, and $N \lhd_c K$, moreover one has the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & S \arrow[r] & \widetilde{G} \arrow[r] & \mathbb{Z}_p \arrow[r] & 1 \\
1 \arrow[r] & N \arrow[r] \arrow[u, hook] & K \arrow[r] \arrow[u, hook] & A \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

where either A is isomorphic to $\mathbb{Z}_p$ or it is trivial. In the latter case $K \cong N$, and K is a Bloch-Kato group. Otherwise, the lower line of the diagram becomes

$$1 \longrightarrow N \longrightarrow K \longrightarrow \mathbb{Z}_p \longrightarrow 1. \tag{5.3}$$

Since $\mathbb{Z}_p$ is a projective pro-p group, (5.3) splits, and we have the isomorphism $K \cong \mathbb{Z}_p \ltimes N$. In particular, $K \cong \mathbb{Z}_p \times N$, since $A \leq \mathrm{Z}(\widetilde{G})$. Therefore, the above argument implies that $H^\bullet(K)$ is a quadratic $\mathbb{F}_p$-algebra, and this proves the theorem. □

On the other hand, direct products of non-abelian free pro-p groups are not Bloch-Kato. For the proof of the next theorem we make use of the following fact, the easy proof of which we leave to the reader.

Fact 5.5. *Let G be a finite p-group, and let M be a finitely generated left $\mathbb{F}_p[G]$-module. Then*

$$\dim_{\mathbb{F}_p}\left(M^G\right) \cdot |C| \geq \dim_{\mathbb{F}_p}(M) \tag{5.4}$$

Theorem 5.6. *Let F_2 be a 2-generated free pro-p group. Then $F_2 \times F_2$ is not Bloch-Kato.*

Proof. Let F_2 be generated by x and r, and consider the presentation

$$1 \longrightarrow R \longrightarrow F_2 \xrightarrow{\varphi} \mathbb{Z}_p \longrightarrow 1,$$

where R is generated as closed normal subgroup by r. We call $\Gamma = F_2 \times_\varphi F_2$ the pullback object of the diagram

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & F_2 \arrow[d, "\varphi"] \\ F_2 \arrow[r, "\varphi"'] & \mathbb{Z}_p \end{tikzcd}$$

in the category of pro-p groups. Namely,

$$\Gamma = F_2 \times_\varphi F_2 = \{(\xi, \xi') \in F_2 \times F_2 \,|\varphi(\xi) = \varphi(\xi')\} \leq_c F_2 \times F_2.$$

In particular, Γ is generated by the pairs (x, x), (r, r) and $(r, 1)$. In fact, every element $(y, y') \in \Gamma$ can be written as

$$(y, y') = (\xi, \xi)(\rho, 1) = (\rho', 1)(\xi, \xi), \tag{5.5}$$

with $\xi \in F_2$ and $\rho, \rho' \in R$.

Let $R^{\mathrm{ab}} = R/[R, R]$ be the abelianization of R, and let $H = R^{\mathrm{ab}} \rtimes \mathbb{Z}_p$, where the action of $\mathbb{Z}_p$ on R^{ab} is induced by the conjugation of F_2. Moreover, let $D \lhd_c \Gamma$ be generated as closed normal subgroup of Γ by (r, r).

By (5.5), the semidirect product $R \rtimes F_2$ maps onto Γ via the map φ', where

$$\varphi'(\rho, \xi) = (\rho, 1)(\xi, \xi), \quad \text{with } \rho \in R,\ \xi \in F_2,$$

which is easily seen to be a homomorphism. Furthermore, φ' is injective, so $R \rtimes F_2$ is isomorphic to Γ.

Suppose that $\varphi'(\rho, \xi) = (\rho, 1)(\xi, \xi) \in D$. Thus $\xi \in R$, which implies that $(\xi, \xi), (\rho, 1) \in D$. By (5.5), one has that every element of D is generated by elements $({}^\xi\rho, \rho)$, with $\rho, \xi \in R$. Furthermore,

$${}^{\xi_1}\rho_1 \cdot {}^{\xi_2}\rho_2 = {}^{\xi_1}(\rho_1\rho_2)y, \quad \text{with } \rho_i, \xi_i \in R, y \in [R, R].$$

Thus, a limit argument shows that $(\rho, 1) \in D$ if, and only if, $\rho \in [R, R]$. This implies that

$$\frac{\Gamma}{D} \cong H = \frac{R \rtimes F_2}{[R,R] \rtimes R}.$$

We want to show that Γ is not finitely presented. Assume for contradiction that it is. Then also H is finitely presented, since D is finitely generated as normal subgroup of Γ.

Claim 1. *The group H is not finitely presented, i.e., $r(H) = \infty$.*

The Hochschild-Lyndon-Serre spectral sequence $H^r(\mathbb{Z}_p, H^s(R^{\mathrm{ab}})) \Rightarrow H^{r+s}(H)$ collapses at the E_2-term, i.e., $E_\infty = E_2$, since $cd(\mathbb{Z}_p) = 1$. Thus one has the following exact sequence:

$$0 \longrightarrow H^1\left(\mathbb{Z}_p, H^1(R^{\mathrm{ab}})\right) \longrightarrow H^2(H) \longrightarrow H^2(R^{\mathrm{ab}})^{\mathbb{Z}_p} \longrightarrow 0. \tag{5.6}$$

Since R is one-generated as normal subgroup, the group R^{ab} is isomorphic to the completed group algebra $\mathbb{Z}_p[\![\mathbb{Z}_p]\!]$. Thus the first cohomology group $H^1(R^{\mathrm{ab}})$ is isomorphic to $(\mathbb{F}_p[\![\mathbb{Z}_p]\!])^*$, and the second cohomology group $H^2(R^{\mathrm{ab}})$ is isomorphic to the second exterior algebra $\Lambda_2(\mathbb{F}_p[\![\mathbb{Z}_p]\!]^*)$. Since $\mathbb{F}_p[\![\mathbb{Z}_p]\!] \cong \varprojlim_k \mathbb{F}_p[\![C_{p^k}]\!]$, where C_{p^k} is the cyclic group of order p^k, one has

$$H^2(R^{\mathrm{ab}})^{\mathbb{Z}}_p \cong \varinjlim_{k,U_k} \Lambda_2\left(\mathbb{F}_p[\![C_{p^k}]\!]^*\right)^{\mathbb{Z}_p/U_k},$$

where $U_k \lhd_o \mathbb{Z}_p$ is such that $\mathbb{Z}_p/U_k \cong C_{p^k}$.

Therefore, Fact 5.5 implies that for all $k \geq 1$ one has

$$\dim_{\mathbb{F}_p}\left(\Lambda_2\left(\mathbb{F}_p[\![C_{p^k}]\!]^*\right)^{\mathbb{Z}_p/U_k}\right) \geq \frac{1}{|\mathbb{Z}_p/U_k|}\dim_{\mathbb{F}_p}\Lambda_2\left(\mathbb{F}_p[\![C_{p^k}]\!]^*\right) = \frac{1}{p^k}\frac{p^k(p^k-1)}{2}.$$

Thus $\dim(H^2(R^{\mathrm{ab}})^{\mathbb{Z}_p}) \geq (p^k - 1)/2$ for all $k \geq 1$, i.e., $H^2(R^{\mathrm{ab}})^{\mathbb{Z}_p}$ has infinite dimension, and so by (5.6) $\dim(H^2(H)) = r(H) = \infty$. This estabilishes the claim.

Therefore the claim implies that Γ is not finitely presented. In particular, $\dim(H^2(\Gamma)) = r(\Gamma) = \infty$, whereas $\dim(H^1(\Gamma)) = d(\Gamma) = 3$. Hence $H^\bullet(\Gamma)$ is not quadratic, and $F_2 \times F_2$ is not Bloch-Kato. □

Remark 5.7. In particular, Theorem 5.6 shows that $F_2 \times F_2$ is not a *coherent* pro-p group, i.e., it contains a finitely generated group which is not finitely presented. By Proposition 4.1, any finitely generated Bloch-Kato pro-p group is a coherent group.

Now Theorem C is the combination of Proposition 5.3, Theorem 5.4 and Theorem 5.6.

Theorem C. *Let G_1 and G_2 be Bloch-Kato pro-p groups, and assume that $G_1 \times G_2$ is Bloch-Kato as well. Then the following restrictions hold:*

(i) *None of G_1 and G_2 is a powerful non-abelian Bloch-Kato group;*
(ii) *at least one of the two groups is abelian.*

In particular, $\mathbb{Z}_p \times S$ is a Bloch-Kato pro-p group for any free pro-p group S.

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